



National
Qualifications
SPECIMEN ONLY

SQ24/AH/01

Mathematics

Date — Not applicable

Duration — 3 hours

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions which contain appropriate working.

State the units for your answer where appropriate.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.



* S Q 2 4 A H 0 1 *

FORMULAE LIST

Standard derivatives	
$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x) dx$
$\sec^2(ax)$	$\frac{1}{a} \tan(ax) + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{x}$	$\ln x + c$
e^{ax}	$\frac{1}{a} e^{ax} + c$

Summations

(Arithmetic series)
$$S_n = \frac{1}{2} n [2a + (n-1)d]$$

(Geometric series)
$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\sum_{r=1}^n r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^n r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r \quad \text{where} \quad \binom{n}{r} = {}^n C_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \dots$$

FORMULAE LIST (continued)

De Moivre's theorem

$$[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta)$$

Vector product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ , about the origin, $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Total marks — 100
Attempt ALL questions

MARKS

1. Given $f(x) = \frac{x-1}{1+x^2}$, show that $f'(x) = \frac{1+2x-x^2}{(1+x^2)^2}$. 3

2. State and simplify the general term in the binomial expansion of $\left(2x - \frac{5}{x^2}\right)^6$.
Hence, or otherwise, find the term independent of x . 3

3. Find $\int \frac{2}{\sqrt{9-16x^2}} dx$. 3

4. Show that the greatest common divisor of 487 and 729 is 1.
Hence find integers x and y such that $487x + 729y = 1$. 4

5. Find $\int x^2 e^{3x} dx$. 5

6. Find the values of the constant k for which the matrix $\begin{pmatrix} 3 & k & 2 \\ 3 & -4 & 2 \\ k & 0 & 1 \end{pmatrix}$ is singular. 4

7. A spherical balloon is being inflated. When the radius is 10 cm the surface area is increasing at a rate of $120\pi \text{ cm}^2 \text{ s}^{-1}$.
Find the rate at which the volume is increasing at this moment. 5

(Volume of sphere = $\frac{4}{3}\pi r^3$, surface area = $4\pi r^2$)

8. (a) Find the Maclaurin expansions up to and including the term in x^3 , simplifying the coefficients as far as possible, for the following: 5

(i) $f(x) = e^{3x}$

(ii) $g(x) = (x+2)^{-2}$

(b) Given that $h(x) = \frac{x e^{3x}}{(x+2)^2}$ use the expansions from (a) to approximate the

value of $h\left(\frac{1}{2}\right)$. 3

9. Three terms of an arithmetic sequence, u_3 , u_7 and u_{16} form the first three terms of a geometric sequence.

Show that $a = \frac{6}{5}d$, where a and d are, respectively, the first term and common difference of the arithmetic sequence with $d \neq 0$.

Hence, or otherwise, find the value of r , the common ratio of the geometric sequence.

4

10. Using logarithmic differentiation, or otherwise, find $\frac{dy}{dx}$ given that

$$e^y = \frac{(3x+2)e^{2x}}{(2x-1)^2}, \quad x > \frac{1}{2}.$$

3

11. Find the exact value of $\int_1^2 \frac{x+4}{(x+1)^2(2x-1)} dx$.

7

12. (a) Given that m and n are positive integers state the negation of the statement:
 m is even or n is even.

1

- (b) By considering the contrapositive of the following statement:

if mn is even then m is even or n is even,

prove that the statement is true for all positive integers m and n .

3

13. Consider the curve in the (x, y) plane defined by the equation $y = \frac{4x-3}{x^2-2x-8}$.

- (a) Identify the vertical asymptotes to this curve and justify your answer.

2

Here are two statements about the curve:

(1) It does not cross or touch the x -axis.

(2) The line $y = 0$ is an asymptote.

- (b) (i) State why statement (1) is false.

(ii) Show that statement (2) is true.

3

14. The lines L_1 and L_2 are given by the following equations.

$$L_1: \frac{x+6}{3} = \frac{y-1}{-1} = \frac{z-2}{2}$$

$$L_2: \frac{x+5}{4} = \frac{y+4}{1} = \frac{z}{4}$$

(a) Show that the lines L_1 and L_2 intersect and state the coordinates of the point of intersection. 5

(b) Find the equation of the plane containing L_1 and L_2 . 3

A third line, L_3 , is given by the equation $\frac{x-1}{2} = \frac{y+7}{4} = \frac{z-3}{-1}$.

(c) Calculate the acute angle between L_3 and the plane. Give your answer in degrees correct to 2 decimal places. 4

15. (a) Given that $f(x) = \ln\left(\frac{1+x}{1-x}\right)$, find $f'(x)$, expressing your answer as a single fraction. 2

(b) Solve the differential equation

$$\cos x \frac{dy}{dx} + y \tan x = \frac{\cos x}{e^{\sec x}}$$

given that $y = 1$ when $x = 2\pi$. Express your answer in the form $y = f(x)$. 7

16. Let $S_n = \sum_{r=1}^n \frac{1}{r(r+1)}$ where n is a positive integer.

(a) Prove that, for all positive integers n , $S_n = \frac{n}{n+1}$. 5

(b) Find

(i) the least value of n such that $S_{n+1} - S_n < \frac{1}{1000}$

(ii) the value of n for which $S_n \times S_{n-1} \times S_{n-2} = S_{n-8}$. 5

17. (a) Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem and the binomial theorem to show that:

$$\cos 4\theta = \cos^4 \theta - 6\cos^2 \theta \sin^2 \theta + \sin^4 \theta$$

and

$$\sin 4\theta = 4\cos^3 \theta \sin \theta - 4\cos \theta \sin^3 \theta.$$

5

- (b) Hence show that $\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$.

3

- (c) Find algebraically the solutions to the equation

$$\tan^4 \theta + 4 \tan^3 \theta - 6 \tan^2 \theta - 4 \tan \theta + 1 = 0$$

in the interval $0 \leq \theta \leq \frac{\pi}{2}$.

3

[END OF SPECIMEN QUESTION PAPER]